LINEAR ALGEBRA HOMEWORK

JULY 28, 2023

Let F be a field and $\mathscr{V}$ be an F-space. A *basis* of $\mathscr{V}$ is a map $v : \mathscr{I} \to \mathscr{V}$ satisfying:

(1) It is independent, *i.e.*, for any $\lambda : \mathscr{I} \to F$ such that $\lambda(i) = 0$ except for finitely many $i \in \mathscr{I}$,
$$\sum_{i \in \mathscr{I}} \lambda(i)v(i) = 0 \Rightarrow \lambda = 0;$$

(2)
$$\begin{aligned} \mathscr{V} &= \sum_{i \in \mathscr{I}} Fv(i) \\ &:= \left\{ \sum_{i \in \mathscr{I}} \lambda(i)v(i) \mid \lambda : \mathscr{I} \to F \text{ zero almost everywhere} \right\}. \end{aligned}$$

Exercise 1. *Let S be a set.* ***Prove*** *that $(F^S, +, \cdot, 0_{F^S})$ defined in today's lecture obeys V1-V8.*

Definition 1. *Let $\left(F^S\right)'$ be the subset of F^S consisting of all maps $S \to F$ which are zero almost everywhere.*

Exercise 2. *Take $\mathscr{I} = S$. Difine*
$$\begin{array}{rccc} v: & S & \longrightarrow & \left(F^S\right)' \\ & s & \longmapsto & \left(\begin{array}{cccc} e_s: & S & \to & F \\ & t & \mapsto & \begin{cases} 0 & \text{if } t \neq s \\ 1 & \text{if } t = s \end{cases} \end{array} \right) \end{array}$$

Prove *that v defines a basis of $\left(F^S\right)'$.*